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HIGHER INTEGRABILITY FOR THE GRADIENT OF SOLUTIONS TO SOME NONLINEAR ELLIPTIC SYSTEMS

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We consider the nonlinear elliptic system $-\operatorname{div}(\mathcal{A}(x, Du(x))) = 0$ where $\mathcal{A}(x, \xi)$ is only hölder continuous with p -growth. We no longer assume differentiability on \mathcal{A} and we prove higher integrability of the gradient Du using fractional Sobolev spaces.

1. Introduction.

Difference quotient technique has been successfully used to prove regularity for weak solutions $u \in W^{1,p}(\Omega)$ of nonlinear elliptic systems

$$(1.1) \quad -\operatorname{div}(\mathcal{A}(x, Du(x))) = 0, \quad x \in \Omega,$$

when $\mathcal{A} = \mathcal{A}(x, \xi)$ is differentiable with respect to x and ξ , [17], [13], [18]. Recently, [16], [10], [9] have dealt with $\mathcal{A} = \mathcal{A}(\xi)$ when no differentiability is assumed. In the present paper we further go on developing the technique

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employed in [16] by allowing \mathcal{A} to depend also on x . Here Ω is a bounded open subset of \mathbb{R}^n , $n \geq 2$, $\mathcal{A} : \Omega \times \mathbb{R}^{n \times N} \rightarrow \mathbb{R}^{n \times N}$ satisfies the following assumptions

$$(1.2) \quad |\mathcal{A}(x, \xi)| \leq c(1 + |\xi|^{p-1}),$$

$$(1.3) \quad |\mathcal{A}(x, \xi) - \mathcal{A}(y, \xi)| \leq c(v + |\xi|)^{p-1}|x - y|^{\theta_1},$$

$$(1.4) \quad |\mathcal{A}(x, \xi) - \mathcal{A}(x, \eta)| \leq c(v + |\xi| + |\eta|)^{p-1-\theta_2}|\xi - \eta|^{\theta_2},$$

$$(1.5) \quad m(v + |\xi| + |\eta|)^{p-2}|\xi - \eta|^2 \leq (\mathcal{A}(x, \xi) - \mathcal{A}(x, \eta))(\xi - \eta),$$

for every $\xi, \eta \in \mathbb{R}^{n \times N}$, for all $x, y \in \Omega$, where $p, c, v, m, \theta_1, \theta_2$ are constants with

$$(1.6) \quad 2 \leq p, \quad 0 < c, \quad 0 \leq v \leq 1, \quad 0 < m, \quad 0 < \theta_1 \leq 1, \quad 0 < \theta_2 \leq 1.$$

In the sequel $u : \Omega \rightarrow \mathbb{R}^N$, $u \in W^{1,p}(\Omega)$ is a weak solution to (1.1), that is

$$(1.7) \quad \int_{\Omega} \mathcal{A}(x, Du(x)) D\phi(x) dx = 0,$$

for every $\phi : \Omega \rightarrow \mathbb{R}^N$, with $\phi \in W_0^{1,p}(\Omega)$. In Section 3 we will prove the following

Theorem. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to (1.1) under (1.2), ..., (1.6). Then*

$$(1.8) \quad Du \in L_{\text{loc}}^r(\Omega), \quad \forall r < p \frac{n}{n-\gamma}, \quad \text{where} \quad \gamma = \min \left\{ 2\theta_1, \frac{2}{2-\theta_2} \right\}.$$

Remark 1. When $\gamma = 2$, then (1.8) holds with $r = p \frac{n}{n-2}$ for $n \geq 3$ and any $r < \infty$ for $n = 2$.

Remark 2. This higher integrability result is achieved by difference quotient technique: hölder continuity of \mathcal{A} allows us to gain only a fractional derivative of $|Du|^{(p-2)/2} Du$ but it is enough in order to improve on the integrability of Du . Fractional Sobolev spaces have been successfully used in [8], [6] for the so-called “natural growth conditions”, in [5] for the nonlinear case $1 < p < 2$, in [7] for the linear case and in [3], [15], [16] when dealing with the so-called “ p, q growth conditions”.

Remark 3. If we consider the integral

$$(1.9) \quad \int_{\Omega} a(x) |Du(x)|^p dx,$$

where $2 \leq p$, $a : \mathbb{R}^n \rightarrow \mathbb{R}$, and

$$(1.10) \quad a \in C^{0,\sigma}(\overline{\Omega}), \quad 0 < \sigma \leq 1, \quad \inf_{\Omega} a > 0,$$

then the Euler equation of the functional (1.9) is (1.7) with $\mathcal{A}(x, \xi) = a(x)p|\xi|^{p-2}\xi$ and such an \mathcal{A} verifies (1.2), ..., (1.6) with $\theta_1 = \sigma$, $\theta_2 = 1$ and $\nu = 0$.

Thus we have the following

Corollary. *If $u \in W^{1,p}(\Omega)$ minimizes the integral (1.9) under (1.10), then*

$$(1.11) \quad Du \in L^r_{\text{loc}}(\Omega), \quad \forall r < p \frac{n}{n-2\sigma}.$$

Remark 4. When $\sigma = 1$, then (1.11) holds with $r = p \frac{n}{n-2}$ and any $r < \infty$ for $n = 2$.

Remark 5. In this paper we deal with the lack of differentiability of $\mathcal{A}(x, \xi)$, but we assume monotonicity (1.5). In [12], [19], [11], [14], they weaken the ellipticity condition (1.5). [19] deals with the existence of solutions while [12], [11], [14] regard their regularity. [12] considers the case $p = 2$ and its ellipticity is weaker than (1.5) but it assumes $\mathcal{A} \in C^1$ and $D\mathcal{A} \in L^\infty$. On the other hand, [11] deals also with $p > 2$, its ellipticity condition is weaker than (1.5) when $\nu > 0$: no degeneration ($\nu = 0$) seems to be allowed in [11]; it assumes $\mathcal{A} \in C^1$ but it removes the growth condition on $D\mathcal{A}$ which was contained in [12]. Eventually, looking at the results, let us point out the difference between our paper and [12], [11], [14]: they prove partial regularity, that is hölder continuity of Du in some open $\Omega_0 \subset \Omega$; thus Du might be discontinuous at some point $x_0 \in \Omega$; in our paper we improve the integrability of Du in the neighbourhood of every point $x_0 \in \Omega$.

2. Preliminaries.

For a vector-valued function $f(x)$, define the difference

$$\tau_{s,h}f(x) = f(x + he_s) - f(x),$$

where $h \in \mathbb{R}$, e_s is the unit vector in the x_s direction, and $s = 1, 2, \dots, n$. For $x_0 \in \mathbb{R}^n$, let $B_R(x_0)$ be the ball centered at x_0 with radius R . We will often suppress x_0 whenever there is no danger of confusion. We now state several lemmas that are crucial to our work. In the following $f : \Omega \rightarrow \mathbb{R}^k$, $k \geq 1$; B_ρ , B_R , $B_{2\rho}$ and B_{2R} are concentric balls.

Lemma 2.1. *If $0 < \rho < R$, $|h| < R - \rho$, $1 \leq t < \infty$, $s \in \{1, \dots, n\}$, f , $D_s f \in L^t(B_R)$, then*

$$\int_{B_\rho} |\tau_{s,h}f(x)|^t dx \leq |h|^t \int_{B_R} |D_s f(x)|^t dx.$$

(See [13], p. 45, [4], p. 28).

Lemma 2.2. *Let $f \in L^t(B_{2\rho})$, $1 < t < \infty$, $s \in \{1, \dots, n\}$; if there exists a positive constant C such that*

$$\int_{B_\rho} |\tau_{s,h}f(x)|^t dx \leq C|h|^t,$$

for every h with $|h| < \rho$, then there exists $D_s f \in L^t(B_\rho)$.

(See [13], p. 45, [4], p. 26).

Lemma 2.3. *If $f \in L^2(B_{3\rho})$ and for some $d \in (0, 1)$ and $C > 0$*

$$\sum_{s=1}^n \int_{B_\rho} |\tau_{s,h}f(x)|^2 dx \leq C|h|^{2d},$$

for every h with $|h| < \rho$, then $f \in L^r(B_{\rho/4})$ for every $r < 2n/(n - 2d)$.

Proof. The previous inequality tells us that $f \in W^{b,2}(B_{\rho/2})$ for every $b < d$, so we can apply the imbedding theorem for fractional Sobolev spaces ([2], Chapter VII).

Lemma 2.4. *For every t with $1 \leq t < \infty$, for every $f \in L^t(B_{2R})$, for every h with $|h| < R$, for every $s = 1, 2, \dots, n$ we have*

$$\int_{B_R} |f(x + he_s)|^t dx \leq \int_{B_{2R}} |f(x)|^t dx.$$

Lemma 2.5. *For every $p \geq 2$*

$$|\tau_{s,h}(|f(x)|^{(p-2)/2} f(x))|^2 \leq k^3 \left(\frac{p}{2}\right)^2 \int_0^1 |f(x) + t \tau_{s,h} f(x)|^{p-2} |\tau_{s,h} f(x)|^2 dt$$

for every $f \in L^p(B_{2R})$, for every h with $|h| < R$, for every $s = 1, 2, \dots, n$, for every $x \in B_R$.

Lemma 2.6. *For every $\gamma > -1$, for every $k \in \mathbb{N}$ there exist positive constants c_2, c_3 such that*

$$(2.1) \quad c_2(|v|^2 + |w|^2)^{\gamma/2} \leq \int_0^1 |v + tw|^\gamma dt \leq c_3(|v|^2 + |w|^2)^{\gamma/2}$$

for every $v, w \in \mathbb{R}^k$.

(See [1]).

The previous Lemma 2.6 allows us to easily get the following corollary.

Corollary 2.7. *For every $p \geq 2$, for every $k \in \mathbb{N}$ there exists a positive constant c_4 such that*

$$(2.2) \quad c_4 \int_0^1 |\lambda + t(\xi - \lambda)|^{p-2} dt \leq (|\lambda| + |\xi|)^{p-2}$$

for every $\lambda, \xi \in \mathbb{R}^k$.

3. Proof of the Theorem.

Let $R > 0$ be such that $\overline{B_{4R}} \subset \Omega$ and let B_ρ and B_R be concentric balls, $0 < \rho < R \leq 1$. Let $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$ be a “cut off” function in $C_0^\infty(B_R)$ with $\eta \equiv 1$ on B_ρ , $0 \leq \eta \leq 1$. Fix $s \in \{1, \dots, n\}$, take $0 < |h| < R$. Using $\phi = \tau_{s,-h}(\eta^2 \tau_{s,h} u)$ in (1.7) we get

$$(3.1) \quad \begin{aligned} (I) &= \int_{B_R} \tau_{s,h}(\mathcal{A}(x, Du)) \eta^2 \tau_{s,h} Du \, dx = \\ &= - \int_{B_R} \tau_{s,h}(\mathcal{A}(x, Du)) 2\eta D\eta \tau_{s,h} u \, dx = (II). \end{aligned}$$

We have

$$(3.2) \quad \begin{aligned} \tau_{s,h}(\mathcal{A}(x, Du(x))) &= \mathcal{A}(x + he_s, Du(x + he_s)) - \mathcal{A}(x, Du(x)) \\ &= \mathcal{A}(x + he_s, Du(x + he_s)) - \mathcal{A}(x, Du(x + he_s)) + \\ &\quad + \mathcal{A}(x, Du(x + he_s)) - \mathcal{A}(x, Du(x)). \end{aligned}$$

In order to estimate (II), we use (3.2), (1.3) and (1.4), thus

$$(3.3) \quad \begin{aligned} |\tau_{s,h}(\mathcal{A}(x, Du(x)))| &\leq c(1 + |Du(x + he_s)|)^{p-1} |h|^{\theta_1} + \\ &\quad + c(v + |Du(x)| + |Du(x + he_s)|)^{p-1-\theta_2} |\tau_{s,h} Du(x)|^{\theta_2}, \end{aligned}$$

then

$$(3.4) \quad \begin{aligned} |(II)| &\leq \int_{B_R} 2\eta(x) |D\eta(x)| |\tau_{s,h} u(x)| c(1 + |Du(x + he_s)|)^{p-1} |h|^{\theta_1} dx \\ &\quad + \int_{B_R} 2\eta(x) |D\eta(x)| |\tau_{s,h} u(x)| c(v + |Du(x)| + \\ &\quad + |Du(x + he_s)|)^{p-1-\theta_2} |\tau_{s,h} Du(x)|^{\theta_2} dx = (III) + (IV). \end{aligned}$$

When dealing with (III), we use the properties of the “cut off” function η , then Hölder inequality, eventually Lemma 2.1 and 2.4:

$$(3.5) \quad \begin{aligned} (III) &\leq c_6 |h|^{\theta_1} \int_{B_R} (1 + |Du(x + he_s)|)^{p-1} |\tau_{s,h} u(x)| dx \leq \\ &\leq c_7 |h|^{\theta_1} \left(\int_{B_R} (1 + |Du(x + he_s)|^p) dx \right)^{\frac{p-1}{p}} \left(\int_{B_R} |\tau_{s,h} u(x)|^p dx \right)^{\frac{1}{p}} \leq \\ &\leq c_8 |h|^{\theta_1} \left(\int_{B_{2R}} (1 + |Du(x)|^p) dx \right)^{\frac{p-1}{p}} \left(\int_{B_{2R}} |D_s u(x)|^p dx \right)^{\frac{1}{p}} |h| \leq c_9 |h|^{\theta_1+1}, \end{aligned}$$

for some positive constants c_6, c_7, c_8, c_9 independent on h . On the other hand, using Young inequality, we deal with (IV) as follows:

$$(3.6) \quad \begin{aligned} (IV) &= \int_{B_R} (\eta(x))^{\theta_2} (v + |Du(x)| + \\ &\quad + |Du(x + he_s)|)^{\frac{(p-2)\theta_2}{2}} |\tau_{s,h} Du(x)|^{\theta_2} 2c |D\eta(x)| (v + |Du(x)| + \\ &\quad + |Du(x + he_s)|)^{p-1-\theta_2-\frac{(p-2)\theta_2}{2}} (\eta(x))^{1-\theta_2} |\tau_{s,h} u(x)| dx \leq \\ &\leq \frac{m}{2} \int_{B_R} (\eta(x))^2 (v + |Du(x)| + |Du(x + he_s)|)^{p-2} |\tau_{s,h} Du(x)|^2 dx + \\ &\quad + c_{10} \int_{B_R} (v + |Du(x)| + |Du(x + he_s)|)^{[p-1-\theta_2-\frac{(p-2)\theta_2}{2}]\frac{2}{2-\theta_2}} |\tau_{s,h} u(x)|^{\frac{2}{2-\theta_2}} dx \\ &= (V) + (VI), \end{aligned}$$

for some positive constant c_{10} independent of h . If $2 < p$ or $\theta_2 < 1$, by Hölder inequality with exponents $\frac{p(2-\theta_2)}{2}$ and $\frac{p(2-\theta_2)}{p(2-\theta_2)-2}$, using Lemma 2.1 and 2.4 we have

$$\begin{aligned} (VI) &\leq c_{10} \left(\int_{B_R} \left(v + |Du(x)| + \right. \right. \\ &\quad \left. \left. + |Du(x + he_s)| \right)^p dx \right)^{\frac{p(2-\theta_2)-2}{p(2-\theta_2)}} \left(\int_{B_R} |\tau_{s,h} u(x)|^p dx \right)^{\frac{2}{p(2-\theta_2)}} \leq \\ &\leq c_{11} \left(\int_{B_{2R}} (1 + |Du(x)|^p) dx \right)^{\frac{p(2-\theta_2)-2}{p(2-\theta_2)}} |h|^{\frac{2}{2-\theta_2}} \left(\int_{B_{2R}} |D_s u(x)|^p dx \right)^{\frac{2}{p(2-\theta_2)}}, \end{aligned}$$

thus

$$(3.7) \quad (VI) \leq c_{12} |h|^{\frac{2}{2-\theta_2}},$$

for some positive constant c_{12} independent on h . When $2 = p$ and $\theta_2 = 1$ we check (3.7) directly. Let us estimate (I) from below. We use (3.2)

$$\begin{aligned} (3.8) \quad (I) &= \int_{B_R} (\mathcal{A}(x + he_s, Du(x + he_s)) - \\ &\quad - \mathcal{A}(x, Du(x + he_s))) \eta^2(x) \tau_{s,h} Du(x) dx + \\ &+ \int_{B_R} (\mathcal{A}(x, Du(x + he_s)) - \mathcal{A}(x, Du(x))) \eta^2(x) \tau_{s,h} Du(x) dx \\ &= (VII) + (VIII). \end{aligned}$$

We apply (1.5) so that

$$\begin{aligned} (3.9) \quad m \int_{B_R} &(v + |Du(x)| + \\ &+ |Du(x + he_s)|)^{p-2} |\tau_{s,h} Du(x)|^2 \eta^2(x) dx \leq (VIII). \end{aligned}$$

In order to deal with (VII), we use (1.3), (1.6), Hölder inequality and

Lemma 2.4:

$$\begin{aligned}
|(VII)| &\leq \int_{B_R} c(\nu + |Du(x)| + |Du(x + he_s)|)^{p-1} |h|^{\theta_1} |\tau_{s,h} Du(x)| \eta^2(x) dx = \\
&= \int_{B_R} (\nu + |Du(x)| + |Du(x + he_s)|)^{\frac{p-2}{2}} \eta(x) |\tau_{s,h} Du(x)| \cdot \\
&\quad \cdot c |h|^{\theta_1} (\nu + |Du(x)| + |Du(x + he_s)|)^{\frac{p}{2}} \eta(x) dx \leq \\
&\leq \left\{ \int_{B_R} (\nu + |Du(x)| + |Du(x + he_s)|)^{p-2} \eta^2(x) |\tau_{s,h} Du(x)|^2 dx \right\}^{\frac{1}{2}} \cdot \\
&\quad \cdot c |h|^{\theta_1} \left\{ \int_{B_R} (\nu + |Du(x)| + |Du(x + he_s)|)^p \eta^2(x) dx \right\}^{\frac{1}{2}} \leq \\
&\leq \frac{m}{4} \int_{B_R} (\nu + |Du(x)| + |Du(x + he_s)|)^{p-2} \eta^2(x) |\tau_{s,h} Du(x)|^2 dx + \\
&\quad + c_{13} |h|^{2\theta_1} \int_{B_R} (1 + |Du(x)|^p + |Du(x + he_s)|^p) dx \leq \\
&\leq \frac{m}{4} \int_{B_R} (\nu + |Du(x)| + |Du(x + he_s)|)^{p-2} \eta^2(x) |\tau_{s,h} Du(x)|^2 dx + \\
&\quad + c_{13} |h|^{2\theta_1} 2 \int_{B_{2R}} (1 + |Du(x)|^p) dx
\end{aligned}$$

thus

$$\begin{aligned}
(3.10) \quad |(VII)| &\leq \frac{m}{4} \int_{B_R} (\nu + |Du(x)| + \\
&\quad + |Du(x + he_s)|)^{p-2} \eta^2(x) |\tau_{s,h} Du(x)|^2 dx + c_{14} |h|^{2\theta_1},
\end{aligned}$$

for some positive constant c_{14} independent on h . Now (3.8), ..., (3.10) merge into

$$\begin{aligned}
(3.11) \quad &\frac{3}{4} m \int_{B_R} (\nu + |Du(x)| + \\
&\quad + |Du(x + he_s)|)^{p-2} |\tau_{s,h} Du(x)|^2 \eta^2(x) dx - c_{14} |h|^{2\theta_1} \leq (I).
\end{aligned}$$

Using (3.1), (3.4), ..., (3.7), (3.11) recalling $0 \leq \nu$, $2\theta_1 \leq \theta_1 + 1$ and $|h| < 1$, we get

$$\begin{aligned}
(3.12) \quad &\frac{m}{4} \int_{B_R} (\eta(x))^2 (|Du(x)| + \\
&\quad + |Du(x + he_s)|)^{p-2} |\tau_{s,h} Du(x)|^2 dx \leq c_{15} |h|^{\min[2\theta_1, \frac{2}{2-\theta_2}]},
\end{aligned}$$

for some positive constant c_{15} independent on h . Now we use Lemma 2.5 and Corollary 2.7 in order to get

$$(3.13) \quad \int_{B_R} |\tau_{s,h}(|Du(x)|^{(p-2)/2} Du(x))|^2 \eta^2(x) dx \leq c_{16} |h|^\gamma,$$

for every $s = 1, \dots, n$, for every h with $|h| < R$, where c_{16} is a positive constant independent on h and γ is defined in (1.8). Since $\eta = 1$ on B_ρ , if $\gamma < 2$ inequality (3.13) allows us to apply Lemma 2.3 in order to get

$$|Du|^{(p-2)/2} Du \in L^r(B_{\rho/4}), \quad \forall r < 2n/(n - \gamma).$$

We remark that $||Du|^{(p-2)/2} Du| = |Du|^{p/2}$, thus (1.8) is completely proven. If $\gamma = 2$ we use (3.13), Lemma 2.2 and Sobolev imbedding theorem for $W^{1,2}$, thus Remark 1 is completely proven. \square

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